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EXISTENCE OF VERTICES OF LOCAL CONNECTIVITY k IN DIGRAPHS OF LARGE OUTDEGREE

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For every positive integer k, there is a positive integer $f(k)$ such that every finite digraph of minimum outdegree $f(k)$ contains vertices x, y joined by k openly disjoint paths.

It was proved in [5] that every finite (undirected) graph (without multiple edges) of minimum degree k contains adjacent vertices joined by k openly disjoint paths (cf. also [6], [7], [1]). It was shown in [8] that every finite digraph (without multiple edges) of minimum outdegree k contains vertices x, y connected by $k-1$ edge-disjoint (directed) paths from x to y, and perhaps there are even k such paths. Such a result is not true for openly disjoint paths in digraphs: for every positive integer m, a finite digraph of minimum outdegree $12m$ is constructed in [8] so that all vertices x, y are connected by at most $11m$ openly disjoint paths from x to y. A similar negative result holds, if we require in addition that also the indegree is large [8]. It remained an open problem in [8], if for every positive integer k there is a (least) integer $f(k)$ such that every finite digraph of minimum outdegree $f(k)$ contains vertices x, y connected by k openly disjoint paths from x to y. We prove the existence of such a function f here and show $f(k)=k$ for $k \leq 3$.

First some definitions and notation. A *digraph* $D=(V,E)$ has no multiple edges of the same direction. Its vertex set is denoted by $V(D)$, its edge set by $E(D) \subseteq \{(x,y): x \neq y \text{ from } V(D)\}$; furthermore, $|D| := |V(D)|$ and $\|D\| := |E(D)|$. The edge (x,y) goes from x to y. For $X \subseteq V(D)$, $D(X)$ means the subdigraph of D spanned by X and $D-X := D(V(D)-X)$; we write $D-x := D-\{x\}$ for $x \in D$, where $x \in D$ is equivalent to $x \in V(D)$. For $x \neq y$ from $V(D)$, $D \cup (x,y) := (V(D), E(D) \cup \{(x,y)\})$; note $D \cup (x,y) = D$, if there is an edge from x to y in D. For $X \subseteq V(D)$, $E_D^-(X) := \{(y,x) \in E(D) : y \notin X \text{ and } x \in X\}$, $N_D^-(X) := \{y \in V(D) : \text{ there is an } (y,x) \in E_D^-(X)\}$ and $d_D^-(X) := |N_D^-(X)|$. We write x instead of $\{x\}$ in this notation and delete the subscript D, if the considered

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D is obvious from the context. The notation $E^+(X)$, $N^+(X)$ and $d^+(X)$ is defined correspondingly; for instance, $d_D^+(x)$ denotes the outdegree of $x \in D$. We call a digraph D *outregular of degree* n, if $d_D^+(x)=n$ for all $x \in D$.

A *path* and a *circuit* in a digraph are always continuously directed and pass through every vertex at most once. An x,y-*path* is a path which starts from the vertex x and ends in the vertex y and an X,y-*path* for $X \subseteq V(D)$ has only the initial vertex and no other vertex in common with X. We call a digraph D *connected*, if there is a (directed!) x,y-path in D for all $x,y \in D$. The maximum number of pairwise openly disjoint x,y-paths in D is denoted by $\kappa(x,y;D)$ for $x \neq y$, and we define $\kappa(x,x;D) := 0$. Furthermore, we set $\overline{\kappa}(D) := \max\limits_{x,y \in D} \kappa(x,y;D)$ for a finite digraph D. A vertex set $T \subseteq V(D)$ *separates* $C \subseteq V(D)$ *to* $x \in D$, if $T \cap (C \cup \{x\}) = \emptyset$ and there is no C, x-path in $D-T$.

If $\mathscr{S}$ is a set of subsets of a set, we write $\bigcup \mathscr{S}$ for $\bigcup\limits_{S \in \mathscr{S}} S$. For a positive integer m, $\mathbb{N}_m := \{1,\dots,m\}$ and $\overleftrightarrow{K}_m := (\mathbb{N}_m, \{(i,j) : i \neq j \text{ from } \mathbb{N}_m\})$.

For non-negative integers p, n, $\mathscr{D}_n^p$ denotes the class of all finite digraphs D which satisfy the inequalities $|\{x \in D : d_D^-(x) \leq n\}| \leq p < |D|$. Every $D \in \mathscr{D}_n^p$ has a vertex x with $d^-(x) > n$, hence $|D| \geq n+2$, since D is a digraph.

Theorem. *For every non-negative integer k, every $D \in \mathscr{D}_{k^3(k+1)}^{k^2(k+1)}$ contains vertices x, y with $\kappa(x,y,D) > k$.*

Proof. We assume that there is a positive integer k, for which this is not true. Be $p := k^2(k+1)$ and $n := k^3(k+1) = kp$. Choose a $D \in \mathscr{D}_n^p$ which has $\kappa(x,y;D) \leq k$ for all $x,y \in D$, such that $|D| + \|D\|$ is as small as possible. Set $P := \{x \in D : d_D^-(x) \leq n\}$. By minimality of D, we have $d_D^-(x) = 0$ for all $x \in P$, $d_D^-(x) = n+1$ for all $x \in D-P$ and $|P| = p$. Choose $x_0 \in V(D) - P \neq \emptyset$ and denote the elements of $E_D^-(x_0)$ by (x_i, x_0) for $i \in \mathbb{N}_{n+1}$. We subdivide (x_i, x_0) by a new vertex s_i for $i \in \mathbb{N}_{n+1}$ ($s_i \neq s_j$ for $i \neq j$) and get in this way D_s; set $S := \{s_i : i \in \mathbb{N}_{n+1}\}$.

By Menger's theorem (see, for instance, chap. X.1 in [10]), for every $x \in D - x_0$ there is a $T_x \subseteq V(D_s) - \{x, x_0\}$ with $|T_x| \leq k$ which separates x to x_0 in D_s. For every $x \in D - x_0$ define $C_x := \{y \in D_s :$ there is an x,y-path in $D_s - T_x\}$. Then $x \in C_x - \bigcup\limits_{y \in D - \{x, x_0\}} C_y$ holds for every $x \in P$, since $d_{D_s}^-(x) = 0$ for $x \in P$. Choose a minimal covering $\mathscr{C} \subseteq \{C_x : x \in D - x_0\}$ of $V(D - x_0)$ and define $T_C := N_{D_s}^+(C)$ for $C \in \mathscr{C}$ and $T := \bigcup\limits_{C \in \mathscr{C}} T_C$. Then $\underline{C} := C - \bigcup(\mathscr{C} - \{C\}) \neq \emptyset$ for every $C \in \mathscr{C}$, since the covering $\mathscr{C}$ of $V(D - x_0)$ is minimal. From the above property of C_x for $x \in P$, we deduce

1. *$C_x \in \mathscr{C}$ and $C_x \cap P = \{x\} = \underline{C_x} \cap P$ for every $x \in P$.*

Furthermore, we have $\underline{C}\cap\underline{C'}=\emptyset$ for $C\neq C'$ from $\mathscr{C}$, $T\cap P=\emptyset$ and

2. $\sum\limits_{C\in\mathscr{C}}|T_C|\leq k|\mathscr{C}|$.

Since $\mathscr{C}$ is a covering of $V(D-x_0)$ and T_C separates C to x_0 in D_s for $C\in\mathscr{C}$, $S\subseteq T$ and $S\cap\bigcup\mathscr{C}=\emptyset$, in particular, $\bigcup\mathscr{C}=V(D-x_0)$ hold. Defining $T'_C:=T_C-S$ for $C\in\mathscr{C}$, we get from (2)

3. $\sum\limits_{C\in\mathscr{C}}|T'_C|\leq k|\mathscr{C}|-n-1$.

Set $\mathscr{C}':=\{C\in\mathscr{C}:\underline{C}\cap P=\emptyset\}$ and $\mathscr{C}_P:=\mathscr{C}-\mathscr{C}'$. By (1), we have $\mathscr{C}_P=\{C\in\mathscr{C}: C\cap P\neq\emptyset\}$ and

4. $|\mathscr{C}_P|=p$.

Consider $\mathscr{A}:=\{A\in\mathscr{C}':\sum\limits_{C\in\mathscr{C}}|\underline{A}\cap T'_C|<k\}$. For $A\in\mathscr{A}$ let $C_1^A,\ \ldots,\ C_{t(A)}^A$ be all $C\in\mathscr{C}$ with $T_C\cap\underline{A}\neq\emptyset$. Then $0\leq t(A)<k$ for every $A\in\mathscr{A}$ by definition of $\mathscr{A}$. For $A\in\mathscr{A}$, define $\overline{A}:=(A\cup\bigcup\limits_{i=1}^{t(A)}C_i^A)-\bigcup(\mathscr{C}-\{A,C_1^A,\ldots,C_{t(A)}^A\})$. Then $\underline{A}\subseteq\overline{A}$ holds for $A\in\mathscr{A}$. Consider any $(x,y)\in E(D_s)$ with $y\in\overline{A}$, but $x\notin\overline{A}\cup\{x_0\}$. Then $(x,y)\in E(D-x_0)$ and there is a $C\in\mathscr{C}-\{A,C_1^A,\ldots,C_{t(A)}^A\}$ containing x. But $y\notin C$, hence $y\in T'_C$. In particular, this implies $N^-_{D-x_0}(\underline{A})\subseteq\overline{A}$, hence $d^-_{D(\overline{A})}(x)\geq n$ for all $x\in\underline{A}$, since $\underline{A}\cap P=\emptyset$ and there is at most one edge from x_0 to x. Therefore, $|\overline{A}|\geq n+1\geq p+1$, since $\underline{A}\neq\emptyset$. Furthermore, $|\mathscr{C}|\geq\frac{n+1}{k}>k$, since $S\subseteq T$. Hence $\overline{A}$ is a proper subset of $V(D-x_0)$, since $|\{A,C_1^A,\ldots,C_{t(A)}^A\}|\leq k$ and $\underline{C}\neq\emptyset$ for every $C\in\mathscr{C}$. Since D was a minimal counterexample, these statements imply that $D(\overline{A}\cup\{x_0\})$ has more then p vertices of indegree at most n. Since every such vertex of indegree at most n must be in $P\cup T\cup\{x_0\}$ by above, we get

5. $|(P\cup T)\cap\overline{A}|\geq p$ *for every* $A\in\mathscr{A}$.

Every $C\in\mathscr{C}$ occurs in $\{C_1^A,\ldots,C_{t(A)}^A\}$ for at most k $A\in\mathscr{A}$, since $|T_C|\leq k$. Hence every $C\in\mathscr{C}$ occurs in $\{A,C_1^A,\ldots,C_{t(A)}^A\}$ for at most $k+1$ $A\in\mathscr{A}$. Every $a\in\overline{A}$ belongs to certain elements of $\{A,C_1^A,\ldots,C_{t(A)}^A\}$, but to no further elements of $\mathscr{C}$. Therefore, every $a\in U(\mathscr{A}):=\bigcup\limits_{A\in\mathscr{A}}\overline{A}$ belongs for at most $k+1$ $A\in\mathscr{A}$ to $\overline{A}$. Hence, (5) implies

6. $|P\cap U(\mathscr{A})|+|T\cap U(\mathscr{A})|=|(P\cup T)\cap U(\mathscr{A})|\geq\frac{p|\mathscr{A}|}{k+1}=k^2|\mathscr{A}|$.

If $C\cap\overline{A}\neq\emptyset$ for some $C\in\mathscr{C}$ and $A\in\mathscr{A}$, then $\underline{C}\subseteq\overline{A}$ by definition of $\overline{A}$. Set $\mathscr{C}'':=\{C\in\mathscr{C}':\underline{C}\subseteq U(\mathscr{A})\}$ and $C'_P:=\{C\in\mathscr{C}_P:\underline{C}\subseteq U(\mathscr{A})\}$. By (1), we get

$|P\cap U(\mathscr{A})|=|\mathscr{C}'_P|$. Since every $C\in\mathscr{C}$ with $\underline{C}\subseteq U(\mathscr{A})$ occurs in $\{A,C_1^A,\dots,C_{t(A)}^A\}$ for some $A\in\mathscr{A}$, we see $|\mathscr{C}''\cup\mathscr{C}'_P|\le k|\mathscr{A}|$. So (6) implies

7. $|T\cap U(\mathscr{A})|\ge k|\mathscr{C}''|+(k-1)|\mathscr{C}'_P|\ge k|\mathscr{C}''|$.

Using (3) and (4), we get from (7)

$$\sum_{C\in\mathscr{C}}|T'_C\cap\bigcup_{C'\in\mathscr{C}'-\mathscr{C}''}\underline{C}'|\le\sum_{C\in\mathscr{C}}|T'_C|-|T\cap U(\mathscr{A})|\le k|\mathscr{C}|-n-1-k|\mathscr{C}''|$$

$$=k|\mathscr{C}_P|+k|\mathscr{C}'-\mathscr{C}''|-n-1<k|\mathscr{C}'-\mathscr{C}''|.$$

But this implies that there is an $A\in\mathscr{C}'-\mathscr{C}''$ with $\sum_{C\in\mathscr{C}}|T'_C\cap\underline{A}|<k$, i.e. $A\in\mathscr{A}$, which contradicts $\mathscr{A}\subseteq\mathscr{C}''$. ∎

Remark. The value p and hence $n=kp$ in the proof is mainly determined by inequality (7): we have to choose p for a given k so large that $|T\cap U(\mathscr{A})|\ge k|\mathscr{C}''|$ holds. Thereby it is enough to consider an $\mathscr{A}_0\subseteq\mathscr{A}$ such that $\mathscr{C}''(\mathscr{A}_0):=\{C\in\mathscr{C}':\underline{C}\subseteq U(\mathscr{A}_0)\}\supseteq\mathscr{A}$ and $|T\cap U(\mathscr{A}_0)|\ge k|\mathscr{C}''(\mathscr{A}_0)|$ hold, where $U(\mathscr{A}_0):=\bigcup_{A\in\mathscr{A}_0}\overline{A}$. Let us choose $A_1,\dots,A_m$ from $\mathscr{A}$ successively so that $A_i\notin\mathscr{C}''(\{A_1,\dots,A_{i-1}\})$ and $\mathscr{C}''(\{A_1,\dots,A_m\})\supseteq\mathscr{A}$ hold; set $\mathscr{A}_0:=\{A_1,\dots,A_m\}$. Suppose $y\in U(\mathscr{A}_0)$ occurs in $\overline{A}$ for $k+1\ge2$ $A\in\mathscr{A}_0$. If $C\in\mathscr{C}$ contains y, then C must be in $\{A,C_1^A,\dots,C_{t(A)}^A\}$ for these A with $y\in\overline{A}$, which implies $C\in\mathscr{A}_0$, since every $C\in\mathscr{C}$ occurs in $\{C_1^A,\dots,C_{t(A)}^A\}$ for at most k $A\in\mathscr{A}$. Therefore, if there are distinct C_1, $C_2\in\mathscr{C}$ containing y, then $C_i\in\mathscr{A}_0$ and $C_{i+1}\in\{C_1^{C_i},\dots,C_{t(C_i)}^{C_i}\}$ for $i=1,2$ (mod 2). But this cannot happen by the choice of $\mathscr{A}_0$. Hence, we conclude $y\in\underline{A}$ for an $A\in\mathscr{A}_0$. But the vertices $x\in\underline{A}$ were not taken into account in deducing inequality (5) for this A, since every $x\in\underline{A}$ has indegree in $D(\overline{A}\cup\{x_0\})$ exceeding n by definition of $\mathscr{A}\subseteq\mathscr{C}'$ and $\overline{A}$. So we see that every $x\in U(\mathscr{A}_0)$ is counted in $|(P\cup T)\cap\overline{A}|\ge p$ for at most k $A\in\mathscr{A}_0$ and we can take $p=k^3$ to get $|T\cap U(\mathscr{A}_0)|\ge k\;|\mathscr{C}''(\mathscr{A}_0)|$. So we have shown that every $D\in\mathscr{D}_{k^4}^{k^3}$ has vertices x,y with $\kappa(x,y;D)>k$.

One can still lessen p by further considerations for all $k\ge2$ (for instance, one can take $p=4$ (13) for $k=2$ (3)), but I do not believe that one can determine $f(k+1)$ in this way. It is not possible to take in our proof p less than k^2, as the case $\mathscr{A}=\{A_1,\dots,A_k\}$, $|A|=t(A)=1$ for $A\in\mathscr{A}$ and $C_1^A=C_1^{A'}$ for $A,A'\in\mathscr{A}$ shows.

From the above remark we get $f(2)=2$. (It is easily shown in a direct way that every finite digraph of minimum indegree 2 contains vertices $x\neq y$ which are joined by 3 openly disjoint paths, two x,y-paths and one y,x-path.) We will show now $f(3)=3$. For this, we need the following well-known property of separating sets.

Lemma. *Let D be a digraph, k a non-negative integer and be $a \in D$ and $X \subseteq V(D-a)$ such that $\kappa(a,x;D) \geq k$ for all $x \in X$. If $C_1 \cap C_2 \cap X \neq \emptyset$ for certain $C_1, C_2 \subseteq V(D-a) - N^+(a)$ with $d^-(C_1) = d^-(C_2) = k$, then also $d^-(C_1 \cap C_2) = d^-(C_1 \cup C_2) = k$ holds.*

For a proof of this result see, for instance, [2], [3] or lemma 1 (3) in [9].

Proposition. *Every $D \in \mathscr{D}_0^0 \cap \mathscr{D}_1^1 \cap \mathscr{D}_2^2 =: \mathscr{D}$ contains x, y with $= \kappa(x,y;D) \geq 3$.*

Proof. We assume that this assertion is not true. Let D be a counterexample such that $|D| + \|D\|$ is as small as possible. Since $D \in \mathscr{D}_2^2$, $|D| \geq 3$. Then D has exactly one vertex of indegree 1, say a, and exactly one vertex of indegree 2, say b, and all the other vertices of indegree 3 by minimality of D. First, we deduce a few properties of D.

1. *D is connected.*

If not, there were a proper, non-empty subset C of $V(D)$ with $E_D^-(C) = \emptyset$. Then $|C| \geq 3$, since $D \in \mathscr{D}_0^0 \cap \mathscr{D}_1^1$, and $D(C)$ were a smaller counterexample.

2. *$\kappa(a,x;D) \geq 2$ for all $x \in D-a$.*

Suppose there is an $x \in D-a$ with $\kappa(a,x;D) = 1$. If $(a,x) \in E(D)$, there is an $X \subseteq V(D-a)$ containing x with $E^-(X) = \{(a,x)\}$. But then $D(X) \in \mathscr{D}$, which contradicts the choice of D. Hence $(a,x) \notin E(D)$, and by Menger's theorem there is a vertex t separating a to x. Define $X := \{y \in D$: there is a y,x-path in $D-t\}$; then $x \in X \subseteq V(D - \{a,t\})$. By (1) there is an X,t-path in D, say, an x_0,t-path. Then $D' := D(X \cup \{t\}) \cup (x_0,t)$ is in $\mathscr{D}$ and has $\overline{\kappa}(D') \leq 2$, which contradicts the choice of D.

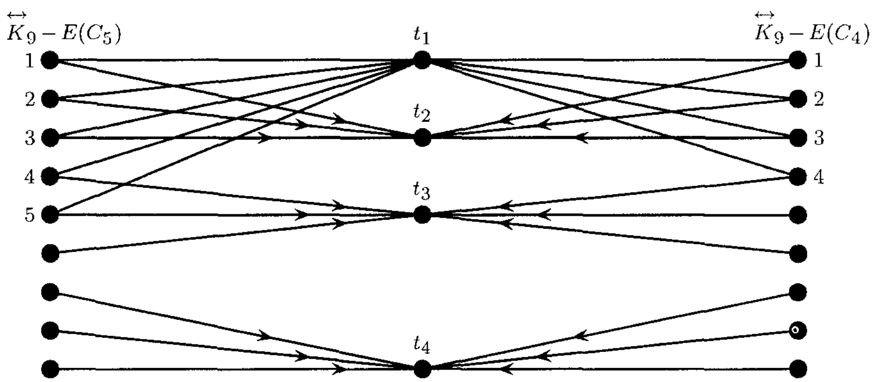
3. *$d^+(a) \geq 3$.*

We suppose $d^+(a) \leq 2$, hence $d^+(a) = 2$ by (2). Be $N^-(a) = \{a'\}$. We have $b \notin N^+(a)$, since otherwise $D - a \in \mathscr{D}$. Furthermore, $(a',x) \in E(D)$ for $x \in N^+(a) - \{a'\}$, since otherwise $D' := (D-a) \cup (a',x) \in \mathscr{D}$ with $\overline{\kappa}(D') \leq 2$. Consider $x \in N^+(a) - \{a'\} \neq \emptyset$. Since $\kappa(a',x;D) \leq 2$, there is no a',x-path in $D' := D - a - (a',x)$. Define $X := \{y \in D' :$ there is a y,x-path in $D'\}$. Hence $x \in X \subseteq V(D' - a')$, in particular, $E_{D-a}^-(X) = \{(a',x)\}$, hence $N^+(a) \cap X = \{x\}$ and $E_D^-(X) = \{(a',x),(a,x)\}$. Since $b \notin N^+(a)$, we have $x \neq b$ and so we get $D(X) \in \mathscr{D}$, which proves (3).

Be $N^+(a) = \{x_1, \ldots, x_n\}$, where $n \geq 3$ by (3). We subdivide (a,x_i) by a new vertex s_i for $i \in \mathbb{N}_n$ ($s_i \neq s_j$ for $i \neq j$) and get so D_s. Set $S := \{s_i : i \in \mathbb{N}_n\}$. Consider $\mathscr{C} := \{C \subseteq V(D-a) : d_{D_s}^-(C) = 2\}$ and let $C_1, \ldots, C_m$ be the maximal elements of $(\mathscr{C}, \subseteq)$. Then $C_1, \ldots, C_m$ form a partition of $V(D-a)$ by definition of D_s, by (2) and $\overline{\kappa}(D) \leq 2$, by Menger's theorem and the above lemma. Set $T_i := N_{D_s}^-(C_i)$ for $i \in \mathbb{N}_m$. Obviously, $S \subseteq \bigcup_{i=1}^{m} T_i$. Since $|S| = n \geq 3$, we may assume $b \notin C_1$ and $T_1 \cap S \neq \emptyset$. Since

$T_1 \subseteq S$ implies $D(C_1) \in \mathscr{D}$, we have $|T_1 \cap S| = 1$, say, $T_1 = \{s_1, t\}$ with $t \notin S$. By (1), there is a C_1,t-path in D, say a c,t-path. Then $D' := D(C_1 \cup t) \cup (c,t) \in \mathscr{D}$ with $\overline{\kappa}(D') \leq 2$, contradicting the choice of D. ∎

The value $f(4)$ is not known. The value $p = 13$ for $k = 3$ mentioned in the remark gives $f(4) \leq 40$. But I do not know, if not perhaps even $f(4) = 4$. The first known k with $f(k) > k$ is $k = 9$. This is shown by the following construction. Take disjoint copies D_1, D_2, D_3, D_4 of the digraph on 22 vertices displayed in the figure, where $C_m := (\mathbb{N}_m, \{(i,i+1) : i \in \mathbb{N}_{m-1}\} \cup \{(m,1)\})$ and an undirected line means a pair of oppositely directed edges. Let $t_j^i \in D_i$ correspond to t_j for $j \in \mathbb{N}_4$ and identify the vertices t_1^i, t_2^{i+1}, t_3^{i+2}, t_4^{i+3} to a vertex t^i for $i = 1,2,3,4 \pmod 4$. The resultant digraph D is outregular of degree 9 and has $\overline{\kappa}(D) = 8$.



In a finite undirected graph of minimum degree n one can always find even *adjacent* vertices joined by n openly disjoint paths. So it is natural to ask, if every finite digraph D of sufficiently large outdegree (dependent on k only) has an edge (x,y) with $\kappa(x,y;D) \geq k$. An answer to this question is not known, but it was shown in [11] that an edge (x,y) with $\kappa(y,x;D) \geq k$ does not necessarily exist. (It is immediate to prove by induction that every $D \in \mathscr{D}_1^1$ has an edge (x,y) with $\kappa(x,y;D) \geq 2$.) One could conjecture that for every k there is an n_k such that every finite digraph D of minimum outdegree n_k contains vertices x,y with $\kappa(x,y;D) \geq k$ and $\kappa(y,x;D) \geq k$. A construction in [8] shows that this is not true even for $k = 2$ and even if, in addition, $\min_{x \in D} d_D^-(x) \geq n_k$.

Another conjecture of mine is related to the problems considered here. It was proved in [4] that every finite undirected graph of minimum degree $n2^{\binom{n}{2}}$ contains a subdivision of the complete graph K_{n+1}. The direct analogue is not true for digraphs after the last paragraph, but perhaps the following holds.

Conjecture. *For every positive integer k, there is a (least) integer $g(k)$ such that every finite digraph of minimum outdegree $g(k)$ contains a subdivision of the transitive tournament of order k.*

Of course, $g(3)=f(2)=2$. But the existence of $g(4)$ as well as a counterexample to $g(4)=3$ are not known.

Added in proof. In the meantime, I proved $g(4)=3$ in "On topological tournaments of order 4 in digraphs of outdegree 3" (to appear in the *Journal of Graph Theory*).

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